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Quantum Dynamical System with Hyperbolic Instabilities

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An analysis of a quantum counterpart of a parametrically kicked nonlinear oscillator is given. The method, using as a basic criterion the recently introduced quantum characteristic exponents, is analogous to the technique developed in classical dynamical system theory. However, our approach to the characterization of the stability of an observable's evolution is done in pure quantum terms.

1. INTRODUCTION

Recently we introduced a new method of studying the Lyapunov instabilities of quantum dynamical systems. Let us explain its basic points.

It seems that the Schrödinger picture is the popular approach to the study of various questions in quantum chaology. However, we shall use the Heisenberg picture. We are motivated by the observation that the Heisenberg picture as well as Heisenberg's equation of motion are in harmonious relation with the very rich algebraic structure of matrix mechanics (or equivalently with the C^*-algebraic structure of the set of observables). This is exactly why within matrix mechanics there is a room for nonlinear dynamical maps without having to face the necessity to generalize quantum mechanics. In contradistinction, the Schrödinger form of the equation of motion does not offer such a possibility. Moreover, the time development of states in this picture can be recast in the form of a unitary, hence linear, transformation. To be more precise, this statement is valid for nonreduced dynamics [cf. the examples with Hartree-type dynamics in Majewski and Kuna (1993a)]. In

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other words, although the equivalence of the Schrödinger and Heisenberg pictures seems to be perfect for all linear problems in quantum mechanics, for questions concerning a proper quantum treatment of stability questions the Heisenberg picture seems to be the appropriate one.

To illustrate this idea let us consider a very simple example. Let us assume that the Hamiltonian H of the system is a function $H = H(A, B)$ of noncommuting dynamical variables A and B. Then, the function

$$A \mapsto e^{iH(A,B)t} A e^{-iH(A,B)t} \tag{1}$$

being a solution of a nonlinear operator differential equation, does not have to be linear while

$$\psi \mapsto e^{-iH(A,B)t}\psi \tag{2}$$

is linear. Moreover, the Heisenberg equation of motion, for $H = H(A, B)$, can be a nonlinear operator equation, although a solution is of course a one-parameter family of linear operators. Therefore, it is natural to examine various stability properties of nonlinear operator functions appearing in the Heisenberg picture.

The aim of this paper is to show that using our method it is possible to examine the stability properties of dynamical maps of type (1) without any modification of the functional form of the fixed Hamiltonian. This will be done by a careful choice of dynamical variables: quadrature operators in our example. In order to avoid any confusion we stress that our choice of dynamical variables is done in such a way that we do not change annihilation and creation operators. In other words we do not violate the quantization procedure (see Section 3). We shall use the algebraic properties of the structure generated by a^* and a.

The paper is organized as follows: in Section 2, a brief description of quantum characteristic exponents is given. The main part of the paper, Section 3, concentrates on a detailed analysis of the quantum counterpart of a parametrically kicked nonlinear oscillator. Our results allow us to say that the presented model is an example of a quantum system with very unstable dynamics for some set of parameters describing the model. Very recently another example of quantum systems with unstable dynamics was given, namely Emch *et al.* (1994) constructed a noncommutative Anosov system.

Finally, let us remark that a quite another characterization of the tendency of noncommutative systems to develop a sort of internal independence can be given in terms of the C^*-algebraic generalization of K–S entropy, K systems, and mixing systems (cf. Benatti, 1993). However, as a noncommutative generalization of the Pesin formula is unknown, a comparison of these two approaches to a description of the random behavior of quantum systems is still impossible. Nonetheless, it should be pointed out that both descriptions are done in the matrix formulation of quantum mechanics.

2. DEFINITIONS

In this section we present a brief account of definitions and properties of quantum exponents. To fix the notation let us recall basic definitions of classical dynamical system theory. A classical dynamical system is defined as a pair $((\Omega, \Sigma, P), (\tau))$ where (Ω, Σ, P) is a probabilistic space, Ω a locally compact, Hausdorff space, and $\{\tau: \Omega \to \Omega\}$ is a measure-preserving transformation. Thus, the (discrete) evolution of the system is given by the iteration of τ. It is well known that if Ω admits a differential calculus, then Lyapunov exponents can be defined as the following limit:

$$\lambda^{cl} = \lim_{n\to\infty} \frac{1}{n} \log|D_x\tau^n(y)| \tag{3}$$

where $D_x\tau^n(y)$ denotes the directional derivatives of τ composed with itself n times at a point x in a direction y (Eckmann and Ruelle, 1985). One can show that a positive Lyapunov exponent measures the average rate of growth of the separation of orbits which at time zero differ by a small vector. This positivity of the Lyapunov exponent can be considered as a basic condition for deterministic chaos in classical dynamical systems.

However, an application of λ^{cl} to physical problems is limited since nature, at least on a microscopic level, is described by quantum laws. To get a proper definition (Majewski and Kuna, 1993a,b) let us replace the probabilistic structure (Ω, Σ, P) in the above definition of a dynamical system by a noncommutative probability space $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a C^*-algebra and ϕ is a state on $\mathcal{A}$. Time evolution will be described by a quantum stochastic map $\tau: \mathcal{A} \to \mathcal{A}$, i.e., a map τ such that:

(i) τ is positive: $\tau(A^*A) \geq 0$ for all $A \in \mathcal{A}$.
(ii) $\tau(1) = 1$.

Let us remark that we do not assume the linearity of τ. Consequently, we defined the quantum counterpart of the dynamical system $(\mathcal{A}, \tau, \phi)$.

For such a system $(\mathcal{A}, \tau, \phi)$ we can define (Majewski and Kuna, 1993a,b)

$$\lambda^q(\tau; x, y) = \lim_{n\to\infty} \frac{1}{n} \log\|(D_x\tau^n)(y)\| \quad (\equiv \lambda^q) \tag{4}$$

where we have used the same notation as in (3), i.e., $D_x\tau^n(y)$ denotes the directional derivatives of τ composed with itself n times at a point x in a direction y. However, now x and y are, in general, noncommutative elements of (C^*-algebra) $\mathcal{A}$. This limit, if it exists, will be called *the quantum characteristic exponent.*

As the first step in a discussion of the basic properties of λ^q it should be shown that λ^q is well defined. We can give an example of *sufficient* conditions for the existence of λ^q. Namely, let us assume:

(i) τ is a completely positive map (in fact, this is the most important assumption).

(ii) $\|\tau^l(0)\| \leq C_1$ for all $l \in N$ and some positive C_1, and

$$\Theta_\tau = \{x \neq 0: \|\tau^l(x)\| \leq C_2\|x\| + \|\tau^l(0)\|\} \neq \varnothing \tag{5}$$

for some positive C_2 and all $l \in N$.

(iii) We have

$$\|D_x\tau^k(y)\| > C^k(x, y)$$

for some positive $C(x, y)$ and all large $k \in N$.

Under the above assumptions one can prove:

Theorem (Majewski and Kuna, 1993a): Let τ: $\mathcal{A} \to \mathcal{A}$ be a map such that the assumptions (i)–(iii) are satisfied. The limit

$$\lim_{n\to\infty} \frac{1}{n} \log\|D_x\tau^n(y)\| \tag{6}$$

exists for $x \in \Theta_\tau$.

Consequently, λ^q is a well-defined notion for the nonempty class of stochastic maps. Let us list the basic properties of the quantum exponents (Majewski and Kuna, 1993a,b):

1. We have

$$\lambda^q(x, y) = \lambda^q(x, ay) \qquad \text{for} \quad a \in \mathbf{R}\backslash\{0\}$$

2. Since the map $y \to D_x\tau^n(y)$ is linear, it is natural to set

$$\lambda^q(x, 0) = -\infty$$

3. Let $\lambda^q(x, y) > \lambda^q(x, z) > -\infty$ and additionally let τ satisfy assumption (iii) in the direction $y + z$. Then

$$\lambda^q(x, y + az) \leq \lambda^q(x, y)$$

for $a \in \mathbf{R}$.

4. The function $y \to \lambda^q(x, y)$ as the limit of continuous functions (in y) is, in general, a Baire function of type I. In particular, the set $\{y \mid y \to \lambda^q(x, y)$ is a continuous function$\}$ is dense in $\mathcal{A}$.

5. Let $\mathcal{A}$ be the C^*-algebra generated by a fixed self-adjoint operator and identity on some Hilbert space, i.e., $\mathcal{A}$ is an Abelian C^*-algebra. Consequently, such a dynamical system $(\mathcal{A}, \tau, \phi)$ should be considered as a classical one. Then, if some mild technical conditions are met (Majewski and Kuna, 1995) the definition (4) leads to the largest classical characteristic exponent.

The above list of properties of λ^q are reminiscent of the basic ones for characteristic exponents (Arnold and Crauel, 1991; Pesin, 1991). Therefore, *we conclude that* λ^q *is the well-defined quantum counterpart of the characteristic exponent.*

3. A MODEL

To provide a physical Hamiltonian model with unstable trajectories of quantum evolution we study the quantum kicked nonlinear oscillator. In particular, we want to show in pure quantum terms that this model exhibits a sensitive dependence on initial conditions, the most important signature of chaos.

Let us recall that although it is easy to give a (mathematical) model with positive quantum characteristic exponent (Majewski and Kuna, 1993a), there still is no model of a purely quantum mechanical system with quantum chaos (Kuna and Majewski, 1993). This is why we reexamine the Milburn model (Milburn, 1990; Milburn and Holmes, 1991; Wielinga and Milburn, 1992). The analysis of this model by Milburn *et al.* shows the presence of signatures of quantum chaos. However, these investigations are not done in pure quantum terms. Therefore the analysis of the model should be supplemented and this is the aim of this section. We shall show that for some values of parameters describing the model, the quantum characteristic exponent λ^q of quadrature components of the electric field for the considered time evolution is positive for one quadrature operator and is negative for the canonically conjugate one. Such behavior, with a slight abuse of language, will be codified in the term *genuine quantum chaos.*

Let us turn to the description of the Milburn model. This model is the quantum optical counterpart of a parametrically kicked nonlinear oscillator. Its Hamiltonian H is of the form (Milburn, 1990; Wielinga and Milburn, 1992)

$$H = H_{\rm NL} + H_{\rm PA} \tag{7}$$

where

$$H_{\rm NL} = \frac{\chi}{2} (a^*)^2 a^2 \tag{8}$$

and

$$H_{PA} = i\hbar \frac{\kappa}{2} [(a^*)^2 - a^2] \sum_{n=-\infty}^{+\infty} \delta(t - n\tau) \tag{9}$$

In above formulas, χ is a constant proportional to the third-order nonlinear susceptibility of the medium, a stands for the boson annihilation operator, κ is the coupling constant (the product of the pump field amplitude and the second-order nonlinearity in the parametric gain medium), and τ is the period of free evolution (i.e., the evolution described by H_{NL}) between each pump pulse. These pump pulses mimick kicks of the harmonic oscillator. The Heisenberg equation of motion for the nonlinear hamiltonian H_{NL} gives the following evolution for a:

$$a(t) = e^{-i\chi t a^* a} a(0) \tag{10}$$

Then, the time evolution of the system can be described (Milburn, 1990; Wielinga and Milburn, 1992) by the equation

$$a(t_n^+) = a(t_n^-)\cosh r + a^*(t_n^-)\sinh r \tag{11}$$

where t_n^+ (t_n^-) is the time just after (before) the passage of the nth pulse, the time dependence of a on t is given by (10), and r is the effective constant for the kick. In the case of a pulsed pump field, r is determined by the integrated time-dependent amplitude of the pump (Milburn, 1990, p. 6568).

Now we want to calculate quantum characteristic exponents λ^q for the evolution $\mathcal{U}$ given by (10) and (11), i.e., $\mathcal{U}a(t_n^+) = a(t_{n+1}^+)$, and a quantum characteristic exponent is defined by

$$\lambda^q(\mathcal{U}; x, y) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log\|(D_x\mathcal{U}^n)(y)\| \tag{12}$$

where $(D_x\mathcal{U}^n)(y)$ denotes the derivatives of $\mathcal{U}$ composed with itself n times at x in direction y. To do so, let us rewrite (10) and (11) in terms of self-adjoint operators Φ and Π, where Φ and Π are defined by

$$a = \Phi + i\Pi \tag{13}$$

Then, (10), (11), and their conjugate equations can be written as

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{-i\mu/2} \begin{pmatrix} e^r \cos \mu B_0 & e^r \sin \mu B_0 \\ -e^{-r} \sin \mu B_0 & e^{-r} \cos \mu B_0 \end{pmatrix} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{14}$$

where $\mu = \chi\tau$ and $B_0 = a^*a - 1/2 = (\Phi^2 + \Pi^2 - 1)$.

Let us remark that (14) gives a nonlinear evolution since the matrix on the right-hand side of (14) is a nonlinear function of *operators* Π and Φ. Now let us change the time evolution (14) slightly. Namely, we impose a control over the possible number of photons in the propagator of free evolution. We ought to do this modification in order to have a real influence of kicks on the mode. Therefore, taking an arbitrary but finite number N, we do not change the physical properties of the considered evolution. In order to carry out the considered change let us denote by p_n the projection on the n-photon state. Clearly

$$[H_{\mathrm{NL}}, P_N] = 0 = [a^*a, P_N]$$

where $P_N = \Sigma_{n=1}^{N} p_n$. Thus we replace (14) by

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{-i\mu/2} \begin{pmatrix} e^r \cos \mu B & e^r \sin \mu B \\ -e^{-r} \sin \mu B & e^{-r} \cos \mu B \end{pmatrix} \begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{15}$$

with $B = (B_0 \cdot P_N) \oplus P_N^\perp$. This procedure *can be legitimated* recalling that real physical experiments deal with light beams with finite number of photons. Next, in order to solve the nonlinear operator equation (15), let us consider the case

$$A^2 \equiv \cos^2 \mu B \cosh^2 r - 1 > \epsilon 1 \tag{16}$$

with arbitrary small $\epsilon > 0$. The opposite case $A^2 < -\epsilon 1$ can be treated in a similar way. Let us observe that the following condition for the parameter μ,

$$\mu \in X = \left\{ x \in R; x > 0, x \cdot \left(n - \frac{1}{2} \right) \neq (2k - 1) \cdot \frac{\pi}{2} \text{ for any } k \in \mathcal{N} \text{ and } n \in N_0 \right\} \tag{17}$$

where $N_0 = \{1, \ldots, N\}$, implies the nontriviality of the operator A ($A \neq -1$). Then, for large enough r, the equality (16) is satisfied. The condition

$$\mu \in Y = \left\{ x \in R; x > 0, x \cdot \left(n - \frac{1}{2} \right) \neq k \cdot \pi \text{ for any } k \in \mathcal{N} \text{ and } n \in N_0 \right\} \tag{18}$$

implies that the function $\sin^{-1} \mu B$ is well defined (note that the spectrum of B is equal to $\{1/2, \ldots (N - 1/2)\}$). Finally,

$$\mu \in Z = \left\{ x \in R; x > 0, x \cdot \left(n - \frac{1}{2} \right) \neq \pm \arccos[(\cosh r)^{-2}] + k\pi \text{ for any } k \in \mathcal{N} \text{ and } n \in N_0 \right\} \tag{19}$$

implies that A^{-1} is well defined. In the sequel, we shall assume that

$$\mu \in X \cap Y \cap Z \tag{20}$$

Under the above assumption we can rewrite (15) as

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{-i\mu/2}\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\begin{pmatrix} \Phi(t_{n-1}^+) \\ \Pi(t_{n-1}^+) \end{pmatrix} \tag{21}$$

where

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -e^{-r}(\sin \mu B)^{-1}(\cos \mu B \sinh r + A) & -e^{-r}(\sin \mu B)^{-1}(\cos \mu B \sinh r - A) \end{pmatrix} \tag{22}$$

$$\mathbf{P}^{-1} = \begin{pmatrix} -\cos \mu B \sinh r\,(2A)^{-1} + \frac{1}{2} & -e^{r} \sin \mu B\,(2A)^{-1} \\ \cos \mu B \sinh r\,(2A)^{-1} + \frac{1}{2} & e^{r} \sin \mu B\,(2A)^{-1} \end{pmatrix} \tag{23}$$

and

$$\mathbf{D} = \begin{pmatrix} \cos \mu B \cosh r - A & 0 \\ 0 & \cos \mu B \cosh r + A \end{pmatrix} \equiv \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \tag{24}$$

Therefore

$$\begin{pmatrix} \Phi(t_n^+) \\ \Pi(t_n^+) \end{pmatrix} = e^{-in\mu/2}\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\begin{pmatrix} \Phi(0) \\ \Pi(0) \end{pmatrix} \tag{21a}$$

where we have used the fact that $\mathbf{D}$ does not depend on t_n. Now we are in a position to study the Lyapunov instabilities of the quadrature components of the electric field during the time evolution given by the formula (21a). To do so, let us define the quadrature operators

$$\Phi^{\epsilon} = \frac{1}{2}\,[e^{i\epsilon}a + e^{-i\epsilon}a^*] \tag{25}$$

and

$$\Pi^{\epsilon} = \frac{1}{2i}\,[e^{i\epsilon}a - e^{-i\epsilon}a^*] \tag{26}$$

The operators Π^{ϵ} and Φ^{ϵ} are related to the amplitude components of the electric field (Yurke, 1989). Clearly

$$(\Phi^\epsilon)^2 + (\Pi^\epsilon)^2 = a^*a + \frac{1}{2} = (\Phi)^2 + (\Pi)^2 \tag{27}$$

Moreover,

$$\begin{pmatrix} \Phi^\epsilon(t_n^+) - \Phi(t_n^+) \\ \Pi^\epsilon(t_n^+) - \Pi(t_n^+) \end{pmatrix} = e^{-in\mu/2}\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\begin{pmatrix} \Phi^\epsilon(0) - \Phi(0) \\ \Pi^\epsilon(0) - \Pi(0) \end{pmatrix} \tag{28}$$

where we have used the invariance of the time evolution equations with respect to the transformation $a \to e^{i\epsilon}a$ [cf. (27)]. To avoid confusion, let us emphasize that (28) does not "smuggle" a linearity into our argument; cf. see the violation of homogeneousness in (28). In other words, we have used the special invariance property (27) for (21a).

Therefore

$$\begin{pmatrix} D_\epsilon(\Phi^\epsilon)^{(n)} \\ D_\epsilon(\Pi^\epsilon)^{(n)} \end{pmatrix} = e^{-in\mu/2}\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\begin{pmatrix} -\Pi(0) \\ \Phi(0) \end{pmatrix} \tag{29}$$

where $D_\epsilon\Phi^\epsilon = \lim_{\epsilon\to 0}[(\Phi^\epsilon - \Phi)/\epsilon](D_\epsilon\Pi^\epsilon = \lim_{\epsilon\to 0}[(\Pi^\epsilon - \Pi)/\epsilon])$.

Remarks. (i) Note that $D_\epsilon\Phi^\epsilon$ is well defined as the derivative with respect to the parameter ϵ of the one-parameter family Φ^ϵ of closed operators. Similarly, $D_\epsilon\Pi^\epsilon$ is related to Π^ϵ.

(ii) The operators Φ^ϵ, Π^ϵ and the phase angle ϵ can be used to characterize squeezed states (Yurke, 1989).

(iii) Let us recall that the basic motivation for the study of characteristic exponents is the problem of stability. Thus, here we study the stability properties of the evolution of quadrature operators in the Milburn model with respect to the phase angle ϵ.

As the final step of calculating a positive quantum exponent for the considered model of the kicked oscillator let us introduce the following cutoff in the (Φ, Π) variables. We replace Φ (Π) by $\Phi_\delta = \int_{-\delta}^{+\delta} \lambda\, dE_\Phi(\lambda)$ $[\Pi_\delta = \int_{-\delta}^{+\delta} \lambda\, dE_\Pi(\lambda)]$, where $\delta \in R^+$, and $\{E_\Phi(\lambda)\}$ $(\{E_\Pi(\lambda)\})$ stands for the spectral resolution of Φ (Π).

Consequently, we will consider

$$\begin{pmatrix} D_\epsilon(\Phi_\delta^\epsilon)^{(n)} \\ D_\epsilon(\Pi_\delta^\epsilon)^{(n)} \end{pmatrix} = e^{-in\mu/2}\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\begin{pmatrix} -\Pi_\delta(0) \\ \Phi_\delta(0) \end{pmatrix} \tag{30}$$

Let us remark that (30) strongly converges to (29) as $\delta \to \infty$. Consequently, (30) is a well-defined approximation of the genuine dynamics.

Then, introducing new variables $\tilde{\Phi}$, $\tilde{\Pi}$

$$\begin{pmatrix} \tilde{\Phi}_\delta \\ \tilde{\Pi}_\delta \end{pmatrix} \stackrel{\text{def}}{=} \mathbf{P}^{-1}\begin{pmatrix} \Phi_\delta \\ \Pi_\delta \end{pmatrix} \tag{31}$$

one has [cf. the second equality in (23)]

$$\|D_\epsilon \tilde{\Phi}^\epsilon_\delta\| = \|\Lambda_1^n \tilde{\Pi}_\delta(0)\| \tag{32}$$

$$\|D_\epsilon \tilde{\Pi}^\epsilon_\delta\| = \|\Lambda_2^n \tilde{\Phi}_\delta(0)\| \tag{33}$$

Further, let us remark that 0 does not belong to the spectrum of Λ_2. Hence, 0 is in the resolvent set of Λ_2 and Λ_2 is a bijection. Therefore,

$$\|\Lambda_2^n \tilde{\Phi}_\delta(0)\| = C\|\Lambda_2^n\| \tag{34}$$

where

$$C = \sup_{g:\, g=\Lambda_2^n f} \|\tilde{\Phi}_\delta(0) g\| \quad (\neq 0) \tag{35}$$

On the other hand, the form of Λ_2 implies

$$\|\Lambda_2^k\| = \|\Lambda_2\|^k \tag{37}$$

for any $k \in \mathcal{N}$. Consequently

$$\begin{aligned} \lambda^q(\tilde{\Pi}_\delta) &= \lim_{k\to\infty} \frac{1}{k} \log\|D_\epsilon \tilde{\Pi}^\epsilon_\delta(t_k)\| \\ &= \lim_{k\to\infty} \frac{1}{k} \log\|\Lambda_2^k\| = \log\|\Lambda_2\| \end{aligned} \tag{38}$$

Moreover, we observe that

$$\begin{aligned} \|\Lambda_1\| &= \sup_{n\in N_0} \Big\{ \left|\cos \mu(n - \tfrac{1}{2}) \cosh r - [\cos^2\mu(n - \tfrac{1}{2}) \cosh^2 r - 1]^{1/2}\right|, \\ &\qquad \left|\cos \mu \cosh r - (\cos^2\mu \cosh^2 r - 1)^{1/2}\right| \Big\} \\ &\equiv \left|\cos \mu(n_o - \tfrac{1}{2}) \cosh r - [\cos^2\mu(n_o - \tfrac{1}{2}) \cosh^2 r - 1]^{1/2}\right| \end{aligned} \tag{39}$$

$$\begin{aligned} \|\Lambda_2\| &= \sup_{n\in N_0} \Big\{ \left|\cos \mu(n - \tfrac{1}{2}) \cosh r + [\cos^2\mu(n - \tfrac{1}{2}) \cosh^2 r - 1]^{1/2}\right|, \\ &\qquad \left|\cos \mu \cosh r + (\cos^2\mu \cosh^2 r - 1)^{1/2}\right| \Big\} \\ &\equiv \left|\cos \mu(n_o - \tfrac{1}{2}) \cosh r + [\cos^2\mu(n_o - \tfrac{1}{2}) \cosh^2 r - 1]^{1/2}\right| \end{aligned} \tag{40}$$

where $n_o \in N_0 \cup \frac{3}{2}$. Then, under our assumptions [see (18)] one can distinguish the following three cases:

$$\text{(i)} \qquad \cos \mu\left(n_o - \frac{1}{2}\right) \cosh r > 1$$

$$\text{(i)} \Rightarrow \|\Lambda_1\| > 1 \quad \text{and} \quad \|\Lambda_2\| < 1$$

$$\text{(ii)} \qquad \cos^2\mu(n_o - \tfrac{1}{2}) \cosh^2 r < 1$$

$$\text{(ii)} \Rightarrow \|\Lambda_1\| = 1 \quad \text{and} \quad \|\Lambda_2\| = 1$$

$$\text{(iii)} \qquad \cos \mu(n_o - \tfrac{1}{2}) \cosh r < -1$$

$$\text{(iii)} \Rightarrow \|\Lambda_1\| < 1 \quad \text{and} \quad \|\Lambda_2\| > 1$$

Now, it is clear that for large enough r [r is the effective constant for kicks introduced in (11)] the norm of Λ_2 is larger than 1. Therefore, we can conclude that for some values of μ, r (i.e., χ, τ, κ) the quantum characteristic exponent λ^q for the quantum variable $\tilde{\Pi}_\delta$ is strictly positive.

4. CONCLUSION

We have shown that, depending on the values of the parameters (χ, τ, κ), the quadrature operators Π^ϵ and Φ^ϵ exhibits chaotic [(i) and (iii)] or regular [(ii)] behavior. In the irregular case [e.g., (iii)] the principal feature of classical chaos—the hyperbolic structure—manifests itself in the existence of a positive characteristic exponent for one canonical coordinate (Π^ϵ) and a negative one for the other (Φ^ϵ). Let us repeat that the operators Π^ϵ and Φ^ϵ are related to the amplitude components of the electric field. Moreover, it can be demonstrated (Yurke, 1989) that a homodyne detector measures these operators. It is also obvious that in polyparametric cases physics as well as mathematics allow numerous combinations of stability in certain directions and irregularity in others. Therefore, we can expect chaotic evolution of the canonically conjugated quadrature operators in our model for some values of χ, τ, κ and regularity for others and we get a confirmation of such behavior. Moreover, let us recall that the analysis done by Milburn suggests similar conclusions. We would like to stress that, in general, our method does not lead to a repetition of semiclassical results (Kuna and Majewski, 1993). The fact that we used some technical assumptions does not mean that we "smuggle" a classical description into our analysis. Let us finish with the conclusion that working within the "pure" quantum description of a genuine nonlinear quantum system, we are able to speak rigorously about hyperbolic instabilities of the quantum evolution without any (semi)classical limits and other (semi)-classical approximations leading to semiquantal models.

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